3. Gladke funkcije in gladke preslikave

Naj bo M neka C^r-mnogoterost in

$\mathcal{U} = \{(U_\alpha, \varphi_\alpha);\ \alpha \in A\}$ atlas.

Definicija: Zvezna funkcija $f : M \to \mathbb{R}$ je C^r-gladka, če velja:

za $\forall \alpha \in A$ je

$$f \circ \varphi_\alpha^{-1} : \underset{\subset \mathbb{R}^n}{\varphi_\alpha(U_\alpha)} \to \mathbb{R}$$

gladka funkcija v običajnem smislu. (Njeni parcialni odvodi do reda r so zvezni).

Trditev: Definicija je dobra.

Dokaz: $x_0 \in M$ $\quad x_0 \in U_\alpha \cap U_\beta$

$$f \circ \varphi_\alpha^{-1} = (f \circ \varphi_\beta^{-1}) \circ (\underbrace{\varphi_\beta \circ \varphi_\alpha^{-1}}_{C^r\text{-difeo. v okolici } \varphi_\alpha(x_0)})$$

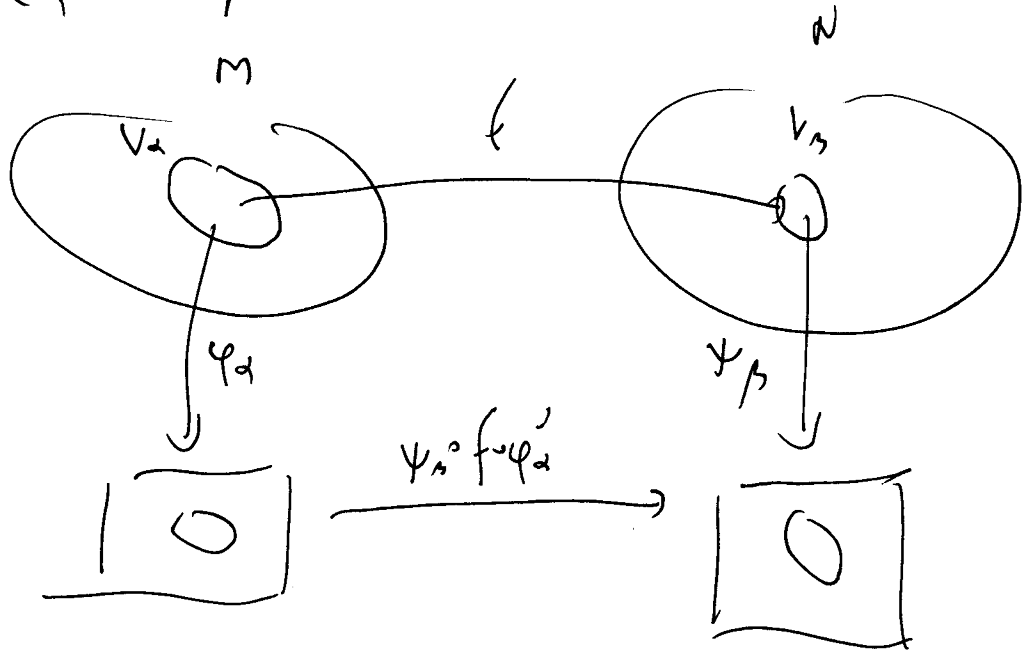
Torej: $f \circ \varphi_\alpha^{-1}$ gladka $\Leftrightarrow$ $f \circ \varphi_\beta^{-1}$ gladka.

Naj bo M C^r-mnt dimenzije m z atlasom $\mathcal{U} = \{(U_\alpha, \varphi_\alpha);\ \alpha \in A\}$ in N C^r-mnt dimenzije n z atlasom $\mathcal{V} = \{(V_\beta, \psi_\beta);\ \beta \in B\}$.

<u>Definicija:</u> Preslikava $f: M \to N$ je C^r-gladka, če je vsak kompozitum

$$\psi_\beta \circ f \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha) \to \psi_\beta(V_\beta)$$

C^r-gladek, ali je nedefiniran.



Trditev: Definicija je dobra.

Dokaz: Dokazati moramo, da je definicija neodvisna od izbire kart: Res:

$$\psi_\beta \circ f \circ \varphi_\alpha^{-1} = (\psi_\beta \circ \psi_{\beta'}^{-1})(\psi_{\beta'} \circ f \circ \varphi_{\alpha'}^{-1}) \circ (\varphi_{\alpha'} \circ \varphi_\alpha^{-1})$$

ker sta $\psi_\beta \circ \psi_{\beta'}^{-1}$ in $\varphi_{\alpha'} \circ \varphi_\alpha^{-1}$ difeomorfizma, je

$\psi_\beta \circ f \circ \varphi_\alpha^{-1}$ gladka $\Leftrightarrow$ $\psi_{\beta'} \circ f \circ \varphi_{\alpha'}^{-1}$ je gladka. $\square$

Še nekaj definicij:

- Lokalni difeomorfizem

  Difeomorfizem

Primer lokalnih difeomorfizmov, ki niso difeom.

$$\pi : S^n \to \mathbb{R}P^n$$
$$x \mapsto [x] = \{x, -x\}$$

in druge krovne preslikave.

III. Tangentni prostor in odvod preslikave

Naj bo najprej $V \subset \mathbb{R}^n$ odprta podmnožica in $p \in V$.

<u>Definicija</u>. Tangentni prostor T_pV mnogoterost V v točki $p \in V$ je množica tangent vseh krivulj, ki gredo skozi p.

$$T_pV = \{ \dot{\gamma}(0) \ ; \ \gamma : (-\varepsilon, \varepsilon) \to V \text{ in } \gamma(0) = p \}$$

To definicijo bomo prilagodili tako, da bo delovala za poljubno abstraktno gladko mnogoterost. Če je naša gladka mnogoterost <u>vložena</u> v $\mathbb{R}^N$, tedaj je zgornja definicija v bistvu že kar prava.

Naj bo M gladka mnogoterost in $\mathcal{U} = \{(U_\alpha, \varphi_\alpha);\ \alpha \in \Lambda\}$ gladek atlas. Spomnimo se, da lokalna karta $(U_\alpha, \varphi_\alpha)$ podmnožico $U_\alpha \subset M$ opremi z lokalnimi koordinatami.

$$\varphi_\alpha : U_\alpha \to V_\alpha \subset \mathbb{R}^n$$

naj bo $\mathbb{R}^n$ opremljen npr. z običajnimi kartezičnimi koordinatami.

$$p \in U_\alpha$$

Koordinate točke p glede na φ_α:

$$(x_1^\alpha(p), x_2^\alpha(p), \dots, x_n^\alpha(p)) = \varphi_\alpha(p)$$

Koordinate so funkcije na M. Lokalno imamo:

$$x_i^\alpha : M \to \mathbb{R}$$

$$x_i^\alpha = x_i \circ \varphi_\alpha$$

$\uparrow$ kartezična koord. na $\mathbb{R}^n$.

<u>Definicija</u> Naj bo $p \in M$ točka v gladki mnogoterosti. Tangentni vektor na M v p je ekvivalenčni razred $[\gamma(t)]$ krivulj $\gamma : (-\varepsilon, \varepsilon) \to M$, $\gamma(0) = p$. Ekvivalenčna relacija v množici krivulj $\{\gamma(t) \ ; \ \gamma(0) = p\}$ je podana s predpisom:

$$\gamma(t) \sim \beta(t) \iff \varphi_\alpha(\dot{\gamma}(t)) = \varphi_\alpha(\dot{\beta}(t)),$$

kjer smo označili

$$\varphi_\alpha(\dot{\gamma}(t)) = \frac{d}{dt}\Big|_{t=0} (\varphi_\alpha(\gamma(t))).$$

Tangentni prostor $T_p M$ je množica vseh ekvivalenčnih razredov $[\gamma(t)]$.

Najprej dokažemo: definicija relacije $\sim$ je dobra, t.j. neodvisna od izbire φ_α.

Naj bo $p \in U_\alpha \cap U_\beta$. Tedaj:

$$\dot{\varphi_\beta}(\gamma) = \frac{d}{dt}\Big|_{t=0} \varphi_\beta(\gamma(t)) = \frac{d}{dt}\Big|_{t=0} (\varphi_\beta \varphi_\alpha^{-1})(\varphi_\alpha(\gamma(t)))$$

$$= D_{\varphi_\alpha(p)}(\varphi_\beta \circ \varphi_\alpha^{-1}) \left(\frac{d}{dt}\Big|_{t=0} \varphi_\alpha(\gamma(t)) \right) \qquad (*)$$

$$= D_{\varphi_\alpha(p)}(\varphi_\beta \circ \varphi_\alpha^{-1}) \left(\dot{\varphi_\alpha}(\gamma) \right)$$

Torej: $\dot{\varphi_\alpha}(\gamma_1) = \dot{\varphi_\alpha}(\gamma_2) \iff \dot{\varphi_\beta}(\gamma_1) = \dot{\varphi_\beta}(\gamma_2)$,

saj je $D_{\varphi_\alpha(p)}(\varphi_\beta \circ \varphi_\alpha^{-1})$ izomorfizem (linearni).

Opazimo še tole: Če je glede na $(U_\alpha, \varphi_\alpha)$ pripadni razredu $[\gamma]$ vektor $\dot{\varphi_\alpha}(\gamma)$, je glede na (U_β, φ_β) ta pripadni vektor

$$\boxed{\dot{\varphi_\beta}(\gamma) = D_{\varphi_\alpha(p)}(\varphi_\beta \circ \varphi_\alpha^{-1}) \left(\dot{\varphi_\alpha}(\gamma) \right)} \quad (TP)$$

transformacijsko pravilo.

Trditev T_pM je vektorski prostor

Dokaz: seštevanje definiramo takole

$$[\gamma_1(t)] + [\gamma_2(t)] = [\varphi_\alpha^{-1}(\varphi_\alpha(p) + t(\dot{\varphi}_\alpha(\gamma_1) + \dot{\varphi}_\alpha(\gamma_2)))]$$

Množenje s skalarjem.

$$\alpha \cdot [\gamma(t)] = [\gamma(\alpha \cdot t)]$$

Dokažimo neodvisnost seštevanja od izbire karte.

Predstavnik $[\gamma_1(t)] + [\gamma_2(t)]$ glede na φ_α:

$$\frac{d}{dt}\Big|_{t=0} \varphi_\alpha(\varphi_\alpha^{-1}(\varphi_\alpha(p) + t(\dot{\varphi}_\alpha(\gamma_1) + \dot{\varphi}_\alpha(\gamma_2))))$$

$$= \frac{d}{dt}\Big|_{t=0} (\varphi_\alpha(p) + t(\dot{\varphi}_\alpha(\gamma_1) + \dot{\varphi}_\alpha(\gamma_2))) =$$

$$= \dot{\varphi}_\alpha(\gamma_1) + \dot{\varphi}_\alpha(\gamma_2)$$

Predstavnik $[\gamma_1(t)] + [\gamma_2(t)]$ glede na φ_β.

$$\frac{d}{dt}\Big|_{t=0} \varphi_\beta\left(\varphi_\alpha^{-1}\left(\varphi_\alpha(p) + t\left(\dot{\varphi}_\alpha(\gamma_1) + \dot{\varphi}_\alpha(\gamma_2)\right)\right)\right)$$

$$= \frac{d}{dt}\Big|_{t=0} (\varphi_\beta \varphi_\alpha^{-1})\left(\varphi_\alpha(p) + t\left(\dot{\varphi}_\alpha(\gamma_1) + \dot{\varphi}_\alpha(\gamma_2)\right)\right)$$

$$= \left(D_{\varphi_\alpha(p)}(\varphi_\beta \varphi_\alpha^{-1})\right)\left(\dot{\varphi}_\alpha(\gamma_1) + \dot{\varphi}_\alpha(\gamma_2)\right).$$

$$= D_{\varphi_\alpha(p)}(\varphi_\beta \varphi_\alpha^{-1})\left(\dot{\varphi}_\alpha(\gamma_1)\right) + D_{\varphi_\alpha(p)}(\varphi_\beta \varphi_\alpha^{-1})\left(\dot{\varphi}_\alpha(\gamma_1)\right)$$

$$\sim \dot{\varphi}_\beta(\gamma_1) + \dot{\varphi}_\beta(\gamma_1),$$

kot smo videli v (*).

<u>Trditev:</u> Naj bo M gladka mnogoterost dimenzije n in $p \in M$. Tedaj je $\dim(T_pM) = n$.

<u>Dokaz:</u> Izberemo $(U_\alpha, \varphi_\alpha)$ in $p \in U_\alpha$.

Preslikava

$$D\varphi_\alpha : T_pM \longrightarrow \mathbb{R}^n$$

$$[\gamma] \longmapsto \dot{\varphi}_\alpha(\gamma)$$

je bijektivna linearna preslikava.

Bijektivnost: po konstrukciji relacije $\sim$. Vsakemu razredu $[\gamma]$ pripada natanko en vektor $\dot{\varphi}_\alpha(\gamma) \in \mathbb{R}^n$.

Linearnost je tudi očitna $\square$

<u>Pomembno:</u> Relacija med reprezentacijama T_pM v $(U_\alpha, \varphi_\alpha)$ in (U_β, φ_β).

$$D\varphi_\alpha : T_pM \longrightarrow \mathbb{R}^n$$

$$[\gamma] \longmapsto \dot{\varphi_\alpha(\gamma)}$$

$$D\varphi_\beta : [\gamma] \longmapsto \dot{\varphi_\beta(\gamma)}$$

$$\dot{\varphi_\beta(\gamma)} = D_{\varphi_\alpha(p)}(\varphi_\beta \circ \varphi_\alpha^{-1})\,(\dot{\varphi_\alpha(\gamma)}) \qquad \text{(TP)}$$

<u>Baza T_pM.</u> Izbira karte $(U_\alpha, \varphi_\alpha)$ nam poda naravno bazo prostora T_pM glede na identifikacijo $D\varphi_\alpha$.

$$\frac{\partial}{\partial x_i^\alpha} = [\varphi_\alpha^{-1}(\varphi_\alpha(p) + t\cdot e_i)] =$$

$$= [\varphi_\alpha^{-1}((p_1^\alpha, p_2^\alpha, \dots, p_n^\alpha) + (0, \dots, 0, \overset{i}{t}, 0, \dots 0))]$$

Tedaj:
$$(D\varphi_\alpha)\left(\frac{\partial}{\partial x_i^\alpha}\right) = e_i$$

Naj bo $\gamma(t)$ poljubna krivulja skozi p v M.

$$\dot{\varphi_\alpha}(\gamma) = \frac{d}{dt}\Big|_{t=0} (\varphi_\alpha(\gamma(t))) =$$

$$= \frac{d}{dt}\Big|_{t=0} (\gamma_1^\alpha(t), \gamma_2^\alpha(t), \ldots, \gamma_n^\alpha(t)) =$$

$$= (\dot\gamma_1^\alpha(0), \dot\gamma_2^\alpha(0), \ldots, \dot\gamma_n^\alpha(0)) =$$

$$= \sum_{i=1}^{n} \dot\gamma_i^\alpha(0) \cdot e_i .$$

Zato:
$$[\gamma] = \sum_{i=1}^{n} \dot\gamma_i^\alpha(0) \frac{\partial}{\partial x_i^\alpha}$$

ker $D\varphi_\alpha([\gamma]) = D\varphi_\alpha\left(\sum_{i=1}^{n} X_i \frac{\partial}{\partial x_i^\alpha}\right) =$

$$\sum_{i=1}^{n} D\varphi_\alpha\left(\frac{\partial}{\partial x_i^\alpha}\right) \cdot X_i = \sum_{i=1}^{n} X_i e_i$$

$$(\Rightarrow) X_i = \dot\gamma_i^\alpha(0)$$

Za vsak tangentni vektor $[\gamma]$ imam transformacijsko pravilo TP oz.:

$$D_{\varphi_\alpha(p)}(\varphi_\beta \varphi_\alpha^{-1})\left(\dot{\varphi_\alpha(\gamma)}\right) = \dot{\varphi_\beta(\gamma)}.$$

Pri tem je

$$\dot{\varphi_\alpha(\gamma)} = D\varphi_\alpha([\gamma]) = (\dot{\gamma}_1^\alpha, \dot{\gamma}_2^\alpha, \ldots, \dot{\gamma}_n^\alpha) \quad \text{in}$$

$$\dot{\varphi_\beta(\gamma)} = D\varphi_\beta([\gamma]) = (\dot{\gamma}_1^\beta, \dot{\gamma}_2^\beta, \ldots, \dot{\gamma}_n^\beta)$$

Preslikava $D_{\varphi_\alpha(p)}(\varphi_\beta \varphi_\alpha^{-1}) : \mathbb{R}^n \to \mathbb{R}^n$ je linearni izomorfizem, za katerega velja:

$$D_{\varphi_\alpha(p)}(\varphi_\beta \varphi_\alpha^{-1}) \begin{pmatrix} \dot{\gamma}_1^\alpha \\ \vdots \\ \dot{\gamma}_n^\alpha \end{pmatrix} = \begin{pmatrix} \dot{\gamma}_1^\beta \\ \vdots \\ \dot{\gamma}_n^\beta \end{pmatrix}$$

Zapišimo sedaj to brez sklicevanja na krivulje.

Elemente $T_p M$ opišemo takole:

Izberemo karto $(U_\alpha, \varphi_\alpha)$ i $p \in U_\alpha$.

Glede na to karto imamo na $T_p M$ naravno bazo $\left\{ \frac{\partial}{\partial x_1^\alpha}, \dots, \frac{\partial}{\partial x_n^\alpha} \right\}$. Elementi v $T_p M$ so torej

$$v = \sum_{i=1}^{n} v_i^\alpha \frac{\partial}{\partial x_i^\alpha}$$

v_i^α – koordinate v glede na karto $(U_\alpha, \varphi_\alpha)$.

V karti (U_β, φ_β) imamo:

$$v = \sum_{i=1}^{n} v_i^\beta \frac{\partial}{\partial x_i^\beta}$$

v_i^β – koordinate glede na karto (U_β, φ_β)

Med koordinatami $\{v_i^\alpha\}$ in $\{v_i^\beta\}$ velja zveza:

$$D_{\varphi_\alpha(p)}(\varphi_\beta \circ \varphi_\alpha^{-1}) \begin{pmatrix} v_1^\alpha \\ v_2^\alpha \\ \vdots \\ v_n^\alpha \end{pmatrix} = \begin{pmatrix} v_1^\beta \\ v_2^\beta \\ \vdots \\ v_n^\beta \end{pmatrix}$$

Tangentni vektorji kot diferencialni operatorji

Smerni odvod funkcij $f: M \to \mathbb{R}$ v točki $p \in M$ vzdolž krivulje $\gamma(t): (-\varepsilon, \varepsilon) \to M$; $\gamma(0) = p$.

$$\frac{d}{dt}\Big|_{t=0} f(\gamma(t))$$

Velja:

$$\frac{d}{dt}\Big|_{t=0} f(\gamma_1(t)) = \frac{d}{dt}\Big|_{t=0} f(\gamma_2(t)),$$

če je $[\gamma_1]_p = [\gamma_2]_p$

Res: $[\gamma_1] = [\gamma_2] \iff \dot{\varphi}_\alpha(\gamma_1) = \dot{\varphi}_\alpha(\gamma_2)$

Imamo pa:

$$\frac{d}{dt}\Big|_{t=0} f(\gamma(t)) = \frac{d}{dt}\Big|_{t=0} f(\varphi_\alpha^{-1}\, \varphi_\alpha \circ (\gamma(t)))$$

$$= \frac{d}{dt}\Big|_{t=0} (f \circ \varphi_\alpha^{-1}) \circ (\varphi_\alpha(\gamma(t))) =$$

$$= D_{\varphi_\alpha(p)}(f \circ \varphi_\alpha^{-1})\,(\dot{\varphi}_\alpha(\gamma))$$

Torej: Pri izbrani $(U_\alpha, \varphi_\alpha)$ je $\frac{d}{dt}\Big|_{t=0} f(\gamma(t))$ odvisen le od $\dot{\varphi}_\alpha(\gamma) \in T_pM$.

Seveda je vse skupaj neodvisno od izbire karte. To je jasno na dva načina: iz zgornjega računa, pa tudi iz:

$$D_{\varphi_\alpha(p)}(f \circ \varphi_\alpha^{-1})\,(\dot{\varphi}_\alpha(\gamma)) = \underbrace{D_{\varphi_\alpha(p)}((f \circ \varphi_\alpha^{-1}) \circ D_{\varphi_\beta(p)}(\varphi_\alpha \circ \varphi_\beta^{-1})}\cdot D_{\varphi_\alpha(p)}(\varphi_\beta \varphi_\alpha^{-1})\,(\dot{\varphi}_\alpha(\gamma))$$

$$= D_{\varphi_\beta(p)}(f \circ \varphi_\beta^{-1})\,(\dot{\varphi}_\beta(\gamma))$$

Naj bo torej $X \in T_pM$ tangentni vektor.

Označimo:

$$(Xf)_{(p)} = \frac{d}{dt}\Big|_{t=0} f(\gamma(t)),$$

za kakršnokoli krivuljo $\gamma : (-\varepsilon, \varepsilon) \to M$; $\gamma(0) = p$,

$$[\gamma] = X.$$

Iz znanih odvodov velja Leibnizovo pravilo.

$$(X(f \cdot g))_p = (Xf)_{(p)} \cdot g(p) + (Xg)_{(p)} f(p) \quad (A)$$

in enako.

$$X(af + bg)_{(p)} = a(Xf)_{(p)} + b(Xg)_{(p)}$$

$$f, g : M \to \mathbb{R}, \quad a, b \in \mathbb{R}, \quad (B)$$

Vsakemu tangentnemu vektorju $X \in T_p M$ pripada natanko določen diferencialni operator I. reda definiran na prostoru zarodkov gladkih funkcij $\mathcal{E}_p^r M$ v točki p.

<u>Velja pa tudi obratno.</u> Naj bo

$$A : \mathcal{E}_p^r \to \mathcal{E}_p^{r-1}$$

linearni operator, za katerega velja:

$$A(f \cdot g) = A(f) \cdot g(p) + A(g) \cdot f(p).$$

Tedaj obstaja natanko določen tangentni vektor $X \in T_p M$, tako da velja:

$$A(f) = (Xf)_{(p)}.$$

Skica dokaza: Najprej ugotovimo, da je dovolj trditev dokazati za pr. $V \overset{odp}{\subset} \mathbb{R}^n$. Nato posplošimo s kartami. (trditev je lokalna - celo infinitezimalna)

Naj bo $$A : \mathcal{E}_p^n \to \mathcal{E}_p^{n-1}$$

Brez škode za splošnost lahko vzamemo $p = 0 \in \mathbb{R}^n$.

Za A velja:
$$A(fg) = A(f)\,g(0) + f(0)\,A(g).$$
$$A(\alpha f + \beta g) = \alpha A(f) + \beta A(g)$$

Od tod:
$$A(c) = 0 \qquad c = \text{konst}.$$

Res:
$$A(c \cdot f) = A(c)\,f(p) + c\,A(f)$$

Zaradi linearnosti:
$$A(cf) = c\,A(f)$$
$$\Rightarrow A(c)\,f(p) = 0 \quad \forall f \Rightarrow A(c) = 0$$

Prav tako velja:

$$A(x_i^2) = 2x_i\big|_0 A(x_i) = 0 \quad \text{za } \forall i.$$

$$A(x_i x_j) = 0$$

Naj bo f poljubna gladka. Razvijmo po Taylorjevi formuli:

$$f(x) = f(0) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} x_i + \sum_{i,j} \frac{\partial^2 f}{\partial x_i \partial x_j} x_i x_j + \dots$$

$$A(f) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} A(x_i)$$

Naj bo $A(x_i) = X_i$.

Torej:

$$A(f) = \sum_{i=1}^{n} X_i \frac{\partial f}{\partial x_i} =$$

$$= \frac{d}{dt}\Big|_{t=0} f(0 + t X_i)$$

□

Odvod preslikave

Povedali smo že, da je odvod preslikave

$$f: \mathbb{R}^n \to \mathbb{R}^m$$

v točki $p \in \mathbb{R}^n$ linearna aproksimacija

$$A: \mathbb{R}^n \to \mathbb{R}^m$$

podana s predpisom:

$$f(p+h) = f(p) + A(h) + o(p,h)$$

$$\lim_{\|h\| \to 0} \frac{\|o(p,h)\|}{\|h\|} = 0$$

$$A = D_p f$$

Prenesimo to idejo na mnogoterosti:

$$f : M \to N$$

Odvod $D_p f$ bo preslikava

$$D_p f : T_p M \to T_{f(p)} N$$

Naravna definicija:

$$[\gamma] \in T_p M$$

naj bo $\quad \gamma(t) : (-\varepsilon, \varepsilon) \to M$

$$\gamma(0) = p$$

predstavnik:

Def:

$$(D_p f)([\gamma]) = [f(\gamma(t))]$$

Izrazimo naš predpis v kartah:

Naj bo $(U_\alpha, \varphi_\alpha)$; $p \in U_\alpha$ in $(V_{\alpha'}, \psi_{\alpha'})$; $f(p) \in V_{\alpha'}$.

Tangentni vektor $[\gamma]$ lahko predstavimo tudi takole:

$$[\gamma] = [\varphi_\alpha^{-1}(\varphi_\alpha(p) + t\,\dot{\varphi_\alpha(\gamma)})] ,$$

kjer je $\dot{\varphi_\alpha(\gamma)} = \frac{d}{dt}\Big|_{t=0} \varphi_\alpha(\gamma(t))$.

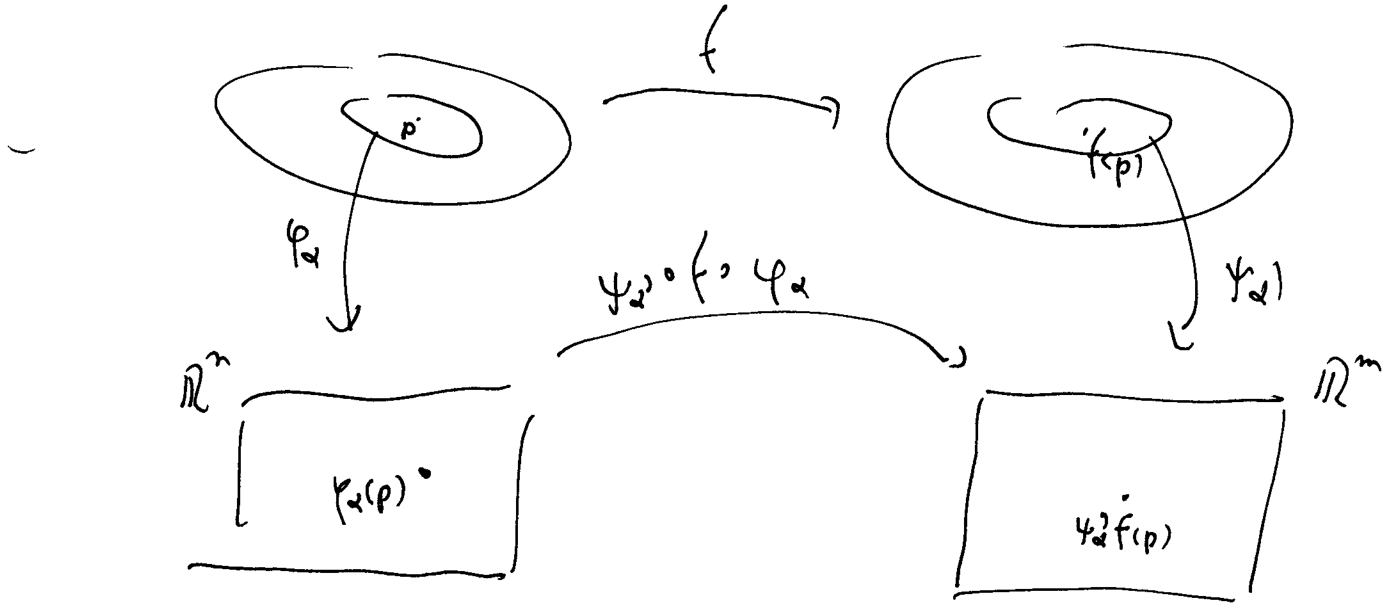
Po definiciji imamo torej:

$$(D_p f)[\gamma] = [(f\,\varphi_\alpha^{-1})(\varphi_\alpha(p) + t\,\dot{\varphi_\alpha(\gamma)})] .$$

Oglejmo si sedaj predstavitev kipa dela $T_{f(p)}N$ v $\mathbb{R}^m$ glede na karto $\psi_{\alpha'}$.

$$\psi_{\alpha'}(\dot D_p f[\xi]) = \frac{d}{dt}\Big|_{t=0} \left(\psi_{\alpha'}\circ f\circ\varphi_\alpha^{-1}\right)\left(\varphi_\alpha(p) + t\,\dot\varphi_\alpha(\xi)\right)$$

$$= D_{\varphi_\alpha(p)}\left(\psi_{\alpha'}\circ f\circ\varphi_\alpha^{-1}\right)\left(\dot\varphi_\alpha(\xi)\right) \qquad (*)$$



Zgoraj je $D_{\varphi_\alpha(p)}(\psi_{\alpha'}\circ f\circ\varphi_\alpha^{-1})$ odvod v običajnem smislu preslikave

$$\psi_{\alpha'}\circ f\circ\varphi_\alpha^{-1} : \mathbb{R}^n \to \mathbb{R}^m$$

Zgornji način lahko zapišemo nekoliko drugače.

Spomnimo se preslikav

$$D_{\varphi_\alpha} : T_p M \to \mathbb{R}^n$$

$$[\gamma] \mapsto \dot{\varphi_\alpha}(\gamma)$$

$$D\psi_{\alpha'} : T_{f(p)} N \to \mathbb{R}^m$$

$$[\beta] \mapsto \dot{\psi_{\alpha'}}(\beta)$$

Tedaj nam $(*)$ pove:

$$\boxed{D_{\psi_{\alpha'}} \circ D_p f \circ D_{\varphi_\alpha}^{-1} = D_{\varphi_\alpha(p)}(\psi_{\alpha'} \circ f \circ \varphi_\alpha)} \quad (*)$$

Preslikava $D_{\varphi_\alpha(p)}(\psi_{\alpha'} \circ f \circ \varphi_\alpha)$ je torej predstavitev $D_p f$ v lokalnih kartah $(U_\alpha, \varphi_\alpha)$, $(V_{\alpha'}, \psi_{\alpha'})$.

$$\boxed{\text{Iz } (*) \text{ vidimo: } D_p f : T_p M \to T_{f(p)} N \text{ - linearna preslikava}}$$

Poglejmo, kaj nam da primerjava predstavitev $D_p f$ v različnih kartah.

Spomnimo se:

$$\dot{\varphi}_\beta(\gamma) = D_{\varphi_\alpha(p)}(\varphi_\beta \varphi_\alpha^{-1})\,(\dot{\varphi}_\alpha(\gamma))$$

Po eni strani imamo:

$$D_{\varphi_\beta(p)}(\psi_{\beta'} f \varphi_\beta^{-1})\,(\dot{\varphi}_\beta(\gamma)) = \psi_{\beta'}(\dot{f(\gamma)}) \qquad (1)$$

Po drugi pa:

$$D_{\varphi_\beta(p)}(\underbrace{\psi_{\beta'}\psi_{\alpha'}^{-1}}\,\underbrace{\psi_{\alpha'} f \varphi_\alpha^{-1}}\,\underbrace{\varphi_\alpha \varphi_\beta^{-1}})\,(\dot{\varphi}_\beta(\gamma)) = \qquad (2)$$

$$= \left(D_{\psi_{\alpha'}(f(p))}(\psi_{\beta'}\psi_{\alpha'}^{-1}) \circ D_{\varphi_\alpha(p)}(\psi_{\alpha'} f \varphi_\alpha^{-1}) \circ D_{\varphi_\beta(p)}(\varphi_\alpha \varphi_\beta^{-1})\right)(\dot{\varphi}_\beta(\gamma))$$

Iz (1) in (2) dobimo:

$$D_{\varphi_\beta(p)}(\psi_{\beta'} f \varphi_\beta^{-1}) = D_{\psi_{\alpha'}(f(p))}(\psi_{\beta'}\psi_{\alpha'}^{-1}) \circ D_{\varphi_\alpha(p)}(\psi_{\alpha'} f \varphi_\alpha^{-1}) \circ$$

$$\circ\, D_{\varphi_\beta(p)}(\varphi_\alpha \varphi_\beta^{-1})$$

Ozirom:

$$D(\psi_\beta f \varphi_\beta^{-1}) = D(\psi_\beta \psi_\alpha^{-1}) \cdot D(\psi_\alpha f \varphi_\alpha^{-1}) \circ (D(\varphi_\beta \varphi_\alpha^{-1}))^{-1}$$

Terminologija:

Prehodni $\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$

$$\mathbb{R}^n \to \mathbb{R}^n$$

pravim prehodna preslikava.

Prehodi $D_{\varphi_\alpha(p)}(\varphi_\beta \varphi_\alpha^{-1})$ pravim tangentna prehodna preslikava, ali pa kratko kar prehodna preslikava.

Konkretno:

$$(\psi_{\alpha'} f \varphi_\alpha^{-1})(\vec{x}) = \begin{pmatrix} f_1^{\alpha'}(x_1^\alpha, \dots, x_n^\alpha) \\ f_2^{\alpha'}(x_1^\alpha \dots x_n^\alpha) \\ \vdots \\ f_m^{\alpha'}(x_1^\alpha \dots x_n^\alpha) \end{pmatrix}$$

$$D_{\varphi_\alpha(p)}(\psi_{\alpha'} f \varphi_\alpha^{-1}) = \begin{pmatrix} \frac{\partial f_1^{\alpha'}}{\partial x_1^\alpha} & \cdots & \frac{\partial f_1^{\alpha'}}{\partial x_n^\alpha} \\ \vdots & & \\ \frac{\partial f_m^{\alpha'}}{\partial x_1^\alpha} & \cdots & \frac{\partial f_m^{\alpha'}}{\partial x_n^\alpha} \end{pmatrix}$$

$m \times n$ - matrika

$$D_{\varphi_\alpha(p)}(\varphi_\beta \varphi_\alpha^{-1}) = \begin{pmatrix} \frac{\partial x_1^\beta}{\partial x_1^\alpha} & \cdots & \frac{\partial x_1^\beta}{\partial x_n^\alpha} \\ \vdots & & \\ \frac{\partial x_n^\beta}{\partial x_1^\alpha} & \cdots & \frac{\partial x_n^\beta}{\partial x_n^\alpha} \end{pmatrix}$$

$n \times n$ obrnljiva matrika.

$$\varphi_\alpha(m) = (x_1^\alpha(m), \ldots, x_n^\alpha(m))$$
$$\varphi_\beta(m) = (x_1^\beta(m), \ldots, x_n^\beta(m))$$

<u>Dve definiciji:</u>

Preslikava $f : M \to N$, ki je odvedljiva v vsaki točki $p \in M$, je submerzija, če je za $\forall p \in M$

$$D_p f : T_p M \to T_{f(p)} N$$

surjektivna preslikava. ($\operatorname{rang} D_p f = \dim N$)

Preslikava $f: M \to N$ je imerzija, če je pri vsakem $p \in M$ preslikava

$$D_p f : T_p M \to T_{f(p)} N$$

injektivna – ima trivialno jedro.

Definicija: Podprostor $N \subset M$ je gladka podmnogoterost, če za vsako točko $p \in N$ obstaja karta $(U_\alpha, \varphi_\alpha)$ mnogoterosti M, taka da velja.

$$\varphi_\alpha(U_\alpha) = V_\alpha$$

$$\varphi_\alpha(U_\alpha \cap N) = V_\alpha \cap \mathbb{R}^k \times \{0\}^{n-k}.$$

Tedaj je N podmnogoterost dimenzije k.

Definicija: Če je $f: N \to M$ gladka in je $f(N) \subset M$ podmnogoterost, ter je f imerzija, je f vložitev.
bijektivna

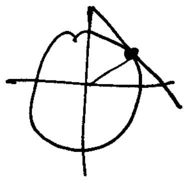
Izračunajmo odvod preslikave $f(t) = (e^{ikt}, e^{ilt})$

$$(D_{t_0} f)\left(\frac{\partial}{\partial t}\right) = \frac{d}{dt}\Big|_{t=t_0} \left(e^{ikt}, e^{ilt}\right) =$$

$$= \left(ik e^{ikt_0}, il e^{ilt_0}\right) = \left(k\, i e^{ikt_0}, l\, i e^{ilt_0}\right).$$

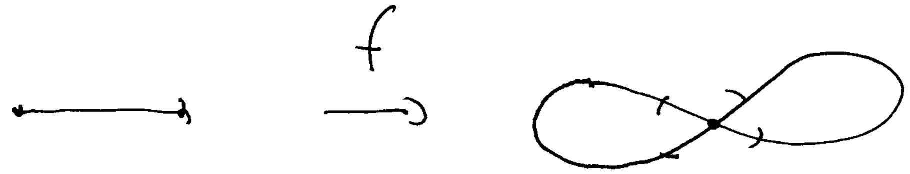
Množenje z i je rotacija za $\frac{\pi}{2}$. t, t_0

e^{ikt_0} in ie^{ikt_0} sta pravokotna vektorja v $\mathbb{C} = \mathbb{R}^2$.

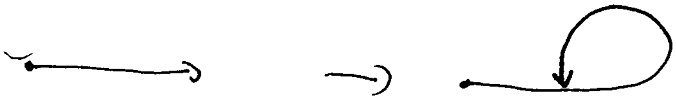


Primeri imerzij, ki niso vložitve.

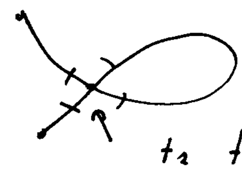
1) $f : [a,b] \to \mathbb{R}^2$



Tudi: Bijektivna imerzija, ki ni vložitev



oz. tudi

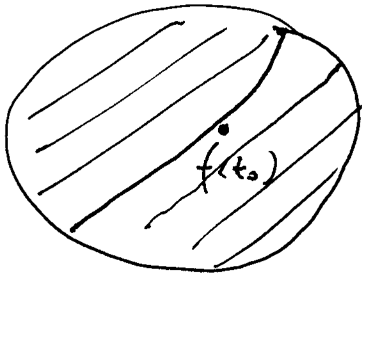


ta točka nima podmnt. karte.

2) $f : \mathbb{R} \to T^2 = \{ (e^{i\varphi}, e^{i\psi}) ;\ \varphi, \psi \in [0, 2\pi) \}$.

$$f(t) = (e^{ikt}, e^{ilt}) \ ; \quad \frac{k}{l} \notin \mathbb{Q}.$$

Preslikava sicer je injektivna, toda točke $p \in f(\mathbb{R})$ nimajo podmnogoterostnih kart.



Izrek: Vsaka imerzija je lokalna vložitev.

Naj bo $f: M \to N$ imerzija in $p \in M$ poljubna točka. Tedaj obstaja $p \in U \overset{odp}{\subset} M$, tako da je

$$f|_U : U \to N$$

vložitev.

Dokaz: Naj bo $\dim(M) = m$ in $\dim(N) = n$, $m \leq n$.

Vemo: $\operatorname{rang} D_p f = m$

Naj bo $(U_\alpha, \varphi_\alpha)$ karta okoli p in (V_β, ψ_β) karta okoli $f(p)$. Linearna preslikava

$$D_{\varphi_\alpha(p)}(\psi_\beta f \varphi_\alpha^{-1}) : \mathbb{R}^m \to \mathbb{R}^n$$

ima rang m. Naj bo $Im = Im\left(D_{\varphi_\alpha(p)}(\psi_\beta f \varphi_\alpha^{-1})\right) \subset \mathbb{R}^n$ slika. To je podprostor dimenzije m v $\mathbb{R}^n$. Naj bo $g \in SO(n)$ rotacija, za katero velja

$$g(Im) = \mathbb{R}^m \times \{0\}^{n-m} \subset \mathbb{R}^n.$$

Naj bo nova karta $(\tilde{V}_\beta, \tilde{\psi}_\beta)$ podana z:

$$\tilde{\psi}_\beta = g \circ \psi_\beta$$

$$\tilde{V}_\beta = g(V_\beta)$$

Zapišimo preslikavo $\tilde{\psi}_\beta \circ f \circ \varphi_\alpha^{-1}$ v koordinatah:

$$F := \tilde{\psi}_\beta \circ f \circ \varphi_\alpha^{-1}$$

$$F(x_1^\alpha, x_2^\alpha, \dots, x_m^\alpha) = \begin{pmatrix} F_1^\beta(x_1^\alpha, \dots, x_m^\alpha) \\ \vdots \\ F_m^\alpha(x_1^\alpha, \dots, x_n^\alpha) \end{pmatrix}$$

Po konstrukciji imamo:

$$D_{\varphi_\alpha(p)} F = \begin{pmatrix} \frac{\partial F_1^\beta}{\partial x_1^\alpha} & \cdots & \frac{\partial F_1^\beta}{\partial x_m^\alpha} \\ \vdots & & \\ \frac{\partial F_n^\beta}{\partial x_1^\alpha} & \cdots & \frac{\partial F_n^\beta}{\partial x_m^\alpha} \end{pmatrix}$$

in

$$R\,DF = \begin{pmatrix} \frac{\partial F_1^\beta}{\partial x_1^\alpha} & \cdots & \frac{\partial F_1^\beta}{\partial x_m^\alpha} \\ \vdots & & \\ \frac{\partial F_m^\beta}{\partial x_1^\alpha} & \cdots & \frac{\partial F_m^\beta}{\partial x_m^\beta} \end{pmatrix}$$

je neizrojena $m \times m$ matrika. (Linearni izomorfizem $\mathbb{R}^m$ na $\mathbb{R}^m \times \{0\}^{n-m}$.

Po izreku o inverzni preslikavi obstaja k okolici $p \in \tilde{\tilde{U}}_\alpha$ in $f(p) \in \tilde{\tilde{V}}_\beta$, tako da je preslikava

$$RF(x_1^\alpha, \dots, x_m^\alpha) = \begin{pmatrix} F_1^\beta(x_1^\alpha, \dots, x_m^\alpha) \\ \vdots \\ F_m^\beta(x_1^\alpha, \dots, x_m^\alpha) \end{pmatrix}$$

difeomorfizem.

Popravimo sedaj karto $(\tilde{V}_\beta, \psi_\beta)$ z malo: naj bo

$$\varphi(y_1^\beta, \dots, y_m^\beta, y_{m+1}^\beta, \dots, y_n^\beta) = (y_1^\beta, \dots, y_m^\beta, y_{m+1}^\beta, \dots, y_n^\beta) -$$
$$- (0, \dots, 0, F_{m+1}^\beta(z), \dots, F_n^\beta(z))$$

Označili smo:

$$z = F^{-1}(y_1^\beta, \dots, y_n^\beta, \dots, y_m^\beta)$$

V lokalni reprezentaciji:

$$\tilde{F} = \varphi \circ F : \tilde{U}_\alpha \longrightarrow \varphi(\tilde{V}_\beta)$$

Imamo sedaj res:

$$\mathrm{Im}(\varphi \circ F) = \varphi(\tilde{U}_\beta) \overset{odp}{\subset} \mathbb{R}^m \times \{0\}^{n-m} \subset \mathbb{R}^n$$

$\square$